**RSPT in 3D**

Now let’s consider RSPT applied to 3D scattering,

**Born Approximation**

The potentials we’ve so far analyzed exactly have been very simple ones. But we’d like to broaden our horizons to more complicated potentials, like the Coulomb potential, etc. To do these exactly would require much technical effort, so instead we’re going to invest in a perturbative approximation. Another advantage of the perturbative approach is that it becomes more effective the larger the energy disparity between our beam (kinetic energy k2/2m) and the well (potential energy ~ V0). So basically high energy beams will give us good results, even at first order. In contrast, when using the spherical harmonic decomposition implicit in the exact approach, high energy beams would require keeping many of the ℓ terms, because it’d only be until we were up to *super* high ℓ that the radial wavefunction’s peaks would be pushed completely out of the potential well, thereby mostly eliminating its phase shift, and thereby its contribution to the scattering cross-section. Vice versa, low energy beams are well approximated by the ℓ = 0 result in the spherical harmonic exact approach whereas we need to keep many terms in the present approaches perturbative expansion to attain a similar level of precision. So the two approaches are complimentary.

We’ll use a variant of the RS perturbation theory on these scattering problems. We start with the Schrodinger equation again,



and we will solve for |ψ> perturbatively, treating V as small. We will (sort of have already) presume that our zeroth order wavefunction is the unscattered one, i.e., the incident beam k>, and we’ll be looking for the wavefunction that develops when that beam hits the target. The latter would include incident and reflected/transmitted beams. Looks like we are implicitly presuming that this state will, by energy conservation, have the same energy, so that’s why we set E = ℏ2k2/2m. And then we also presume that the state our |k> scatters into is an eigenstate of the total Hamiltonian. This could be so if we imagine that the scattering potential is slowly turned on, as it were, because then, by the adiabatic theorem, the state our |k> develops into would be the corresponding eigenstate of the full H. Okay well now, to get a perturbative series, we append the parameter λ to the perturbation,



and then expand |ψ> in powers of λ and solve at each order,



The first equation is:



and its solution is:



i.e., the incoming current beam wavefunction, where |A|2 is the current density and k is the particle wavenumber. There are no other terms in this expression for ψ0(r) because at this order there is no potential, and thus no possibility of reflection or transmission. OK, now let’s look at the next order term,



So this is the solution to first order term. The term



is called the ‘operator’ Green’s function (will explain the +iε, where ε is infinitesimal, to the denominator, in a minute). And it’s called ‘on-shell’ because E = ℏ2k2/2m, as we’ll recall. We will want to know what G0 is in position space. So let’s work out,



To evaluate the (triple) integral what we do first is align our pz axis with the vector (**r**-**r**′), which simplifies the problem and enables the felicitous introduction of spherical coordinates,



At this point it’s a good idea to change variables to **p′** = ћ**k′**. Then we have:



where k is our incoming particles’ wavenumber. One more integral to do. OK, now we may notice that this integral is singular – because there are two singularities: one each at k′ = ± k. Despite the fact that this integral cannot be done as stated, it doesn’t mean we should give up. There are various ways to handle the singularities. We can calculate the principle value of the integral, or we can shift the denominator above or below the real k′ axis by a small complex amount. There isn’t necessarily a way to know which way it should be done until you do it. We will want to do it the way that results in something which we’ll be able to relate to our asymptotic scattering form. And it turns out that shifting the denominator below the real k′ axis a bit is what we should do. This procedure results in traveling waves. So we will write this as:



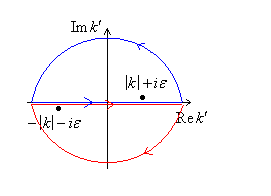
in which case the poles are at:



And then to prepare to do this integral using residues we’ll write the sin using Euler’s identity. Then we have:



We’ll integrate the integrand around a closed semi-circular contour in the upper half plane for the first integral. It must be in the u.h.p. in order to converge since it is here that eik|**r**-**r**′| goes to zero. Conversely, we’ll close the contour down for the second integral (red contour).



The integral along the semi-circular contours are zero since the integrand goes to zero there. And so the residue theorem tells us that:



and,



So altogether, subtracting the two up we get:



So there we have it:



Anyway, we can proceed with our perturbative series. The next (second) order term is:



So it seems pretty clear that generally we have:



The last equation is a recursive equation which you can easily verify is accurate by plugging in the prior expression for |ψ> and checking that it works out. As we did in 1D, we can introduce the ‘on shell’ T-matrix – remember E is fixed to some value Ek = k2/2m.



And can also introduce the total GF,



and write everything in terms of it,



**Relating T-matrix to the scattering cross section**

Now we’re ready to get the scattering cross section. We’ll take our perturbative solution of ψ, project onto position space, and then extract the asymptotic scattering form from it. Thus we’ll have identified f(θ,φ) and from there we can get the cross section. So starting with,



and projecting onto position space,



Now observe that the **r**′ integral runs only over the region where V(**r**′) is non-zero, which is typically a finite region in space. We’re interested in the case where r is large and we can say much larger than r′. So in that case we have:



and so we can make the approximation,



where in the last line we define , which is the direction of the solid angle dΩ. Comparing to the scattering form introduced in the 3D scattering file,



we see that:



which can be written, for convenience, as:



where |k> is a momentum eigenstate (normalized to delta function) Now let’s fill in the expansion of the wavefunction,



to get:



Dividing both sides by A, and recalling from the 3D scattering file,



we have:



Let’s do a quick unit check. We’ll note that |p> has units of 1/√(momentum)3 since <p|p´> is normalized to δ3(p-p´), which has units of 1/(momentum)3.



So that works out. There’s another nice way to write this. Note that,



But now recall from above that the fully developed, perturbed wavefunction is:



So we can say,



where |> = (1 + G0V + G0VG0V + ….)|**k**> is what |**k**> turns into when we slowly turn on the perturbation. And therefore,



And could put the tilde on the k´ instead too.

**Relating T-matrix to phase shift(s)**

FWIW, comparing with the exact result



we see that we can write Tkʹk as:



where cosθ = **k**·**k**´/k2. Well the negative 1 isn’t really anticipated yet – we could’ve put just 1 or any pure phase there – but that’s what we’ll need as it turns out. It follows that the Legendre transform component is of Tk´k is:



where we used the normalization condition of the Legendre polynomials (see angular momentum eigenfunctions) So we can say,



and, kind of obviously, all of it,



Anyway, here is a good place to put the **optical theorem**. Recall we found in the 3D scattering file, that:



Let’s compare this with the imaginary part of Tkk (this is Tk´k(cos0))



And so we can see that the total cross section is given by:



Note since (can see this by comparing formulas for dσ/dΩ in terms of T and f, though again negative sign maybe not anticipated, but necessary)



(θ is angle between **k** and **k**´) This is often written as:



Most people don’t have A’s in their formulas, but that’s because they don’t define the incoming wavefunction with an A – but I want to keep the A so that our setup is same as in 1D.

**Relating T-matrix to S-matrix**

We can relate this to the S-matrix, which in |kℓm> space is just e2iδ(ℓ,k) along the diagonal [Tℓk, Tℓk, notation whatever].



So, since Sℓk = e2iδ\_ℓk, we have:



If we were to go back to **k-**space, then we should have:



So we have:

